2023. 3.21

(Szemerédi Regularity lemma)

Given $\varepsilon > 0$, and $m \in \mathbb{N}$, there exists $M = M(\varepsilon, m)$. such that any graph G admits an ε-regular partition $V = V_0 \cup V_1 \cdots \cup V_r$ with $m \le r \le M$.

(Triangle removal lemma)

$\forall \varepsilon > 0$, $\exists \delta > 0$, such that for any graph G on n vertices with at most δn^3 triangles, it may be made triangle-free by removing at most εn^2 edges.

G: ε-far from triang free if one has to remove at least εn^2 edges from G in order to make it triangle-free.

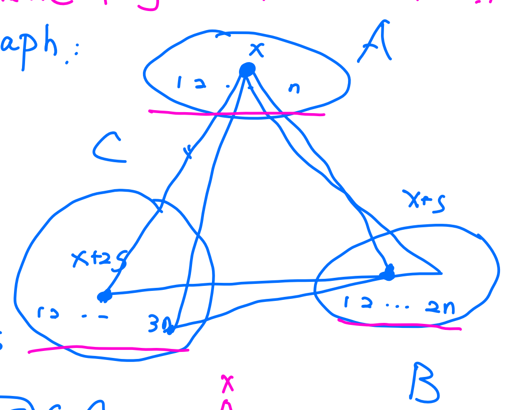
$(\longleftarrow)$ $\forall \varepsilon > 0$, $\exists$ $\delta = \delta(\varepsilon) > 0$ such that if G is a ε-far from being triangle free that G has at least δn^3 triangles

Theorem (Roth) $\forall \delta > 0$, there exists n_0 such that for $n \ge n_0$, any subset A of $[n]$ with at least δn elements contains an arithmetic progression of length 3. (3-AP) x $x+d$ $x+2d$

$x + y = 2z$

(If $S \subseteq [n]$ does not contain 3-term arithmetic progression, then $|S| = o(n)$)

proof. Given S we construct the following graph:



G has $6n$ vertices in three sets A, B, C.

$|A| = n$. $A = [n]$ $B = [2n]$ $C = [3n]$

We treat the vertices as integers

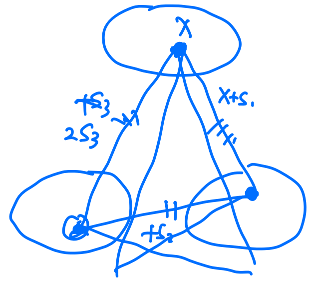
$\forall x \in [n]$, $\forall s \in S$, we put a triangle $T_{x,s}$ in G.

$T_{x,s}$: $x \in A$, $x+s \in B$, $x+2s \in C$ $\quad$ $\underset{x+2s\quad x+s}{\overset{x}{\Delta}}$

$n \cdot |S|$ We claim that the $n|S| = \delta n^2$ triangles we put into G are edge disjoint. As this graph has δn^2 edge-disjoint triangles, it is clear that at least δn^2 edges needs to be removed in order to make it triangle free. By the triangle removal lemma, We get that G has at least $c(\delta)n^3$ triangles. For n sufficiently large, $c(\delta)n^3 > \delta n^2$, hence we get a "new" triangle which is not one of those we explicitly constructed.

In other words, we get a triangle,

So that the labels on these three edges are (S_1, S_2, S_3). Then



$$x+S_1+S_2 = x+2S_3 \qquad S_1+S_2=2S_3$$

that is, S_1, S_3, S_2 form a (non-trivial) 3-term AP.

(Szemerédi's theorem) ~~for δ~~ 3-AP $\to$ k-AP.

$\boxed{\text{regularity lemma in hypergraph}}$ } Gowers; Nagle, Rödl, Schacht & Skokan.

Proof of Erdős-Stone-Simonovits Theorem.

H: $\chi(H)=r$ $\quad \boxed{ex(n,H) \leqslant \left(1-\frac{1}{r-1}+\varepsilon\right)\times\frac{n^2}{2}}$ $\quad \left(1-\frac{1}{\chi(H)-1}+\varepsilon\right)\times\frac{n^2}{2}$

n sufficiently large.

1. partition
2. cleaning
3. Counting/embedding

Let H be a graph with t vertices, $\chi(H)=r$, maximum degree Δ.
Suppose that G is a graph on n vertices with at least $\left(1-\frac{1}{r-1}+\varepsilon\right)\times\frac{n^2}{2}$ edges. We will embed H into G.

Let $V(G)=X_1\cup X_2\cup\cdots\cup X_m$ be a $\frac{1}{2}\left(\frac{\varepsilon}{8}\right)^{\Delta}\Delta^{-1}$ (ε') regular partition of the vertex set of G. We remove edges as previous: removing xy if

① $(x,y)\in X_i\times X_j$, where (X_i,X_j) is not ε'-regular

② $(x,y)\in X_i\times X_j$ where $d(X_i,X_j)<\frac{\varepsilon}{4}$

③ $x \in X_i$, where $|X_i| < \frac{\varepsilon}{16m} n$.

The number of edges removed in ① is at most $\sum_{(i,j) \in I} |X_i||X_j| \le \frac{\varepsilon' \cdot n^2}{\frac{1}{2} \times \left(\frac{\varepsilon}{8}\right)^{\Delta} \cdot \Delta^{-1} \cdot n^2} \le \frac{1}{16} \times \varepsilon \times n^2$

The number of edges removed in ② is at most $\frac{\varepsilon}{4} n^2$ $\quad \sum X_i \quad \frac{\varepsilon}{4} \times (|X_1| + |X_2| + \cdots + |X_M|) \le \frac{\varepsilon}{4} \cdot n^2$

The number of edges removed in ③ is at most. $\quad \frac{\varepsilon}{16m} \cdot n \times n \times m = \frac{\varepsilon}{16} n^2$

$|X_i| < \frac{\varepsilon}{16m} \cdot n$

removed $\le \frac{\varepsilon}{16} n^2 + \frac{\varepsilon}{4} n^2 + \frac{\varepsilon}{16} n^2 = \frac{3}{8} \varepsilon n^2$. We have removed at most $\frac{3}{8} \varepsilon n^2$ edges. Hence the graph G' that remains after all these edges have been removed has density at least $\left(1 - \frac{1}{r-1} + \frac{\varepsilon}{8}\right)$

It must contain a copy of K_r. $\quad \left(1 - \frac{1}{r-1} + \varepsilon\right) \times \frac{n^2}{2}$

We may suppose that this lies between sets $V_1, V_2, \cdots V_r$.

Because of our removal process, $|V_j| \ge \frac{\varepsilon}{16m} \cdot n \ge 2\varepsilon^{-\Delta} t$

the graph between V_i and V_j has density at least $\frac{\varepsilon}{4}$, and is $\ge 2\varepsilon''$

$\varepsilon' = \frac{1}{2} \left(\frac{\varepsilon}{8}\right)^{\Delta} \cdot \Delta^{-1}$ - regular.

By "embedding lemma", we can embed H into G.

Lemma 1. Let $\varepsilon > 0$, Let G be a graph and suppose that $V_1, V_2, \cdots V_r$ are subsets of $V(G)$, such that $|V_i| \ge 2\varepsilon^{-\Delta} t$ for each $1 \le i \le r$ and the graph between V_i and V_j has density $d(V_i, V_j) \ge 2\varepsilon$ $\frac{\varepsilon}{4}$ and

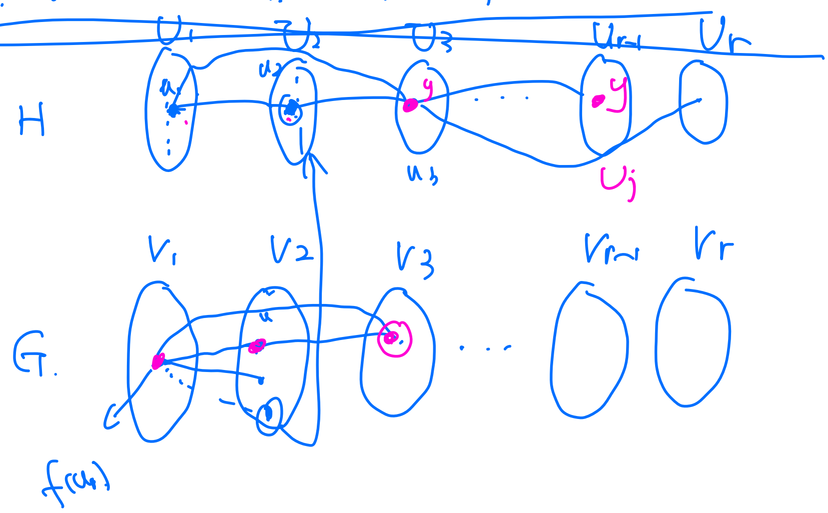
$\frac{1}{2}\varepsilon^{\Delta}\Delta^{-1}$-regular for all $1 \le i < j \le r$. Then G contains a copy of any graph H with t vertices. $\chi(H) = r$ and $\Delta(H) = \Delta$.

$\forall d > 0$, and Δ, $\exists\ \varepsilon = \varepsilon(d, \Delta)$ and $c = c(d, \Delta)$ so that the following holds. Assume $V_1 \cdots V_r$ are vertex sets of size at least ch such that every i, j the pair (V_i, V_j) is ε-regular and $d(V_i, V_j) \ge d$. Then. if H is a graph on at most h vertices with $\chi(H) \le r$ and $\Delta(H) \le \Delta$. then $V_1, V_2, \cdots, V_r$ span a copy of H.

proof of Lemma 1.

$\chi(H) \le r$

$\Delta(H) \le \Delta$

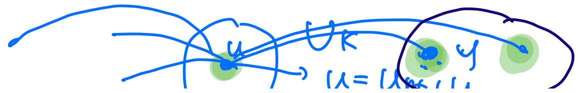


We will give an embedding f of H into G so that $f(U_i) \subset V_i$ $\forall 1 \le i \le r$.

Let the vertices of H be $u_1, u_2, \cdots, u_t$. For each $1 \le h \le t$, let $L_h = \{u_1, u_2, \cdots, u_h\}$. $\forall y \in U_j \setminus L_h$. let T_y^h be the set of vertices in V_j which are adjacent to all already embedded neighbor of y. $N_h(y) = N(y) \cap L_h$. T_y^h is the set of vertices in V_j adjacent to every element of $f(N_h(y))$

We will prove that $\forall y \in V(H) \setminus \underbrace{L_h}_{u_1, u_2 \cdots u_h}$. $|T_y^h| \ge \varepsilon^{|N_h(y)|} \cdot |V_j|$ by induction on h.

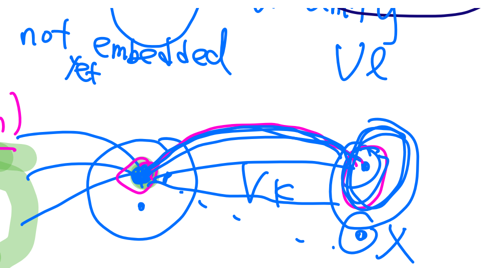
$h = 0$ ✓. We may assume that L_h has been embedded consistent with the induction hypothesis and attempt to embed $u = u_{h+1} \in U_k$ into an appropriate $v \in T_u^h$.



Let Y be the set of neighbors of u which are not embedded

We wish to find an element $v \in T_u^h \setminus f(L_h)$

such that $\forall y \in Y$ $|N(v) \cap T_y^h| \geq \varepsilon |T_y^h|$



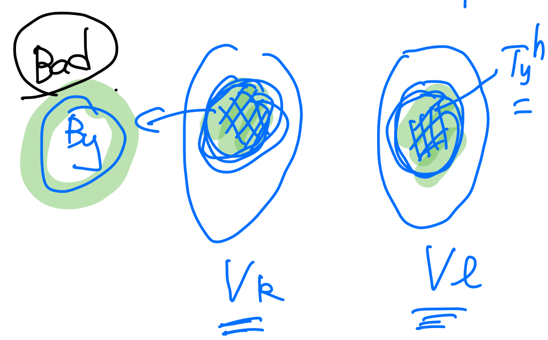
If such a vertex exists, taking $f(u) = v$ and $T_y^{h+1} = N(v) \cap T_y^h$

Let B_y the sets of vertices in T_u^h which are "bad" for $y \in Y$.

that is, $|N(v) \cap T_y^h| < \varepsilon \cdot |T_y^h|$

By induction, $|T_y^h| \geq \varepsilon^\Delta |V_\ell|$

$(y \in V_\ell)$



We must have $|B_y| < \frac{1}{2} \varepsilon^\Delta \Delta^{-1} |V_k|$

Otherwise the density between B_y and T_y^h would be less than ε, contradiction the regularity assumption on G.

Since $|V_k| \geq 2 \varepsilon^{-\Delta} t$.

$$|T_u^h \setminus \bigcup_{y \in Y} B_y| \geq \varepsilon^\Delta \cdot |V_k| - \frac{1}{2} \varepsilon^\Delta \cdot \Delta^{-1} |V_k| \times \Delta$$

$$\geq \varepsilon^\Delta |V_k| - \frac{1}{2} \varepsilon^\Delta \cdot |V_k|$$

$$\geq \frac{1}{2} \varepsilon^\Delta \cdot |V_k| \geq \frac{1}{2} \times \varepsilon^\Delta \cdot \varepsilon^{-\Delta} \cdot t \times 2$$

$$\geq t$$

Since at most $t-1$ vertices have already been embedded, we have a choice for $f(u)$.

$\boxed{\text{Triangle-removal lemma}}$ triangle counting lemma

① ↳ ↓ K_4

$\boxed{\text{Graph removal lemma}}$ graph counting lemma

② Graph regularity lemma $\longrightarrow$ hypergraph regularity lemma.

③ "supersaturation"